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Current-induced noise and damping in nonuniform ferromagnets

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In the presence of spatial variation in the magnetization direction, electric current noise causes a fluctuating spin-transfer torque that increases the fluctuations of the ferromagnetic order parameter. By the fluctuation-dissipation theorem, the fluctuations at thermal equilibrium are related to the viscous magnetization damping, which in nonuniform ferromagnets acquires a nonlocal tensor structure. At the hand of spin spirals, we demonstrate that the current-induced noise and damping increase with the gradient of the magnetization texture and becomes significant for narrow domain walls.

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Three decades ago, Berger[1,2] showed that an electric current passing through a ferromagnetic domain wall exerts a torque on the wall. The spin of the electron that traverses the wall adiabatically adapts to the local exchange field, thereby transferring angular momentum to the magnetization. Subsequently, it was realized that the same effect also exists in magnetic multilayers.[3] Sufficiently strong current-induced torques switch the magnetization direction in multilayers or move domain walls in bulk magnets. The early ideas have been confirmed both theoretically and experimentally.[4]

Recently, the importance of noise for current-induced magnetization dynamics has drawn attention. Although often noise is undesired, it may in some cases be quite useful. Wetzels *et al.*[5] showed that current-induced magnetization reversal of spin valves is accelerated by an increased level of current noise. The noisy current exerts a fluctuating torque on the magnetization.[6] Ravelosona *et al.*[7] reported observation of thermally assisted depinning of a narrow domain wall under an applied current. Thermally assisted current-driven domain-wall motion has also been studied theoretically.[8,9]

The present paper addresses current-induced magnetization noise in nonuniformly magnetized ferromagnets. The spatial variation in the magnetization direction gives rise to increased magnetization noise; by a fluctuating spin-transfer torque, electric current noise causes fluctuations of the magnetic order parameter. The increased magnetization noise can be represented by introducing fictitious stochastic magnetic fields. By the fluctuation-dissipation theorem (FDT), the thermal stochastic field is related to the dissipation of energy, and thus the damping of the magnetization dynamics. Since the correlator of the stochastic field in general is inhomogeneous and anisotropic, the damping is a nonlocal tensor. Ferromagnetic spin spirals are interesting model systems to study these effects since the field correlator and damping become spatially independent. It is shown that for spirals with relatively short wavelength (~20 nm), the current-induced noise and damping is substantial. We consider here disordered metallic ferromagnets in which the scattering mean-free path is smaller than the spatial scale of the ferromagnet. This implies that a spin spiral is a good model for a domain wall with equal magnetization gradient at its center. We therefore conclude that current-induced magnetization noise and damping should be an issue for sufficiently narrow domain walls.

It is instructive to start with an introduction to the FDT for *uniform* (single-domain) ferromagnetic systems, characterized by a time-dependent unit magnetization vector $\mathbf{m}(t)$ and saturation magnetization magnitude M_s. The spontaneous equilibrium noise of such *macrospins* is described by the correlator $S_{ij}(t-t')=\langle \delta m_i(t)\delta m_j(t')\rangle$, where $\delta\mathbf{m}(t)=\mathbf{m}(t)-\langle\mathbf{m}(t)\rangle$ is the random deviation of the magnetization from the mean value at time t. The brackets denote statistical averaging at equilibrium, and i and j Cartesian components. Applying an external magnetic field $\mathbf{h}^{(\text{ext})}(t)$, the magnetization can be excited from the equilibrium state. For a sufficiently weak perturbation, the resulting change in magnetization is

$$\Delta m_i(t) = \sum_j \int dt' \chi_{ij}(t-t') h_j^{(\text{ext})}(t'), \tag{1}$$

defining the magnetic susceptibility $\chi_{ij}(t-t')$ as the causal response function. In the present model we consider only the transverse response. The FDT relates this susceptibility to the equilibrium noise correlator:[10]

$$S_{ij}(t-t') = \frac{k_B T}{M_s V}\int d\omega e^{-i\omega(t-t')}\frac{\chi_{ij}(\omega)-\chi_{ji}^*(\omega)}{i2\pi\omega}, \tag{2}$$

where T is the temperature and V is the volume of the ferromagnet. Alternatively, the FDT can be expressed in terms of a fictitious random magnetic field $\mathbf{h}(t)$ with zero mean, which is regarded as the cause of the fluctuations $\delta\mathbf{m}(t)$. Noting that Eq. (1) implies that $\delta m_i(\omega)=\Sigma_j \chi_{ij}(\omega)h_j(\omega)$ in frequency space, it follows from Eq. (2) that

$$\langle h_i(t)h_j(t')\rangle = \frac{k_B T}{M_s V}\int d\omega e^{-i\omega(t-t')}\frac{[\chi_{ji}^{-1}(\omega)]^*-\chi_{ij}^{-1}(\omega)}{i2\pi\omega}, \tag{3}$$

where $\chi_{ij}^{-1}(\omega)$ is the ij component of the Fourier-transformed inverse susceptibility tensor.

The magnetic susceptibility can be found from the Landau-Lifshitz-Gilbert (LLG) equation of motion,

$$\frac{d\mathbf{m}}{dt} = -\gamma\mathbf{m}\times[\mathbf{H}_{\text{eff}}+\mathbf{h}^{(\text{ext})}]+\alpha_0\mathbf{m}\times\frac{d\mathbf{m}}{dt}. \tag{4}$$

Here γ is the gyromagnetic ratio, $\mathbf{h}^{(\text{ext})}(t)$ is the weak excitation introduced in Eq. (1), α_0 is the Gilbert damping constant, and $\mathbf{H}_{\text{eff}}$ is an effective magnetic field that includes a static magnetic field, magnetic anisotropies, and dipolar fields. Linearizing this equation in the magnetic response to $\mathbf{h}^{(\text{ext})}(t)$, we find the inverse susceptibility

$$\chi^{-1} = \frac{1}{\gamma}\begin{bmatrix} \gamma|\mathbf{H}_{\text{eff}}| - i\omega\alpha_0 & i\omega \\ -i\omega & \gamma|\mathbf{H}_{\text{eff}}| - i\omega\alpha_0 \end{bmatrix} \tag{5}$$

written in matrix (tensor) form in the plane normal to the equilibrium magnetization direction. Inserting Eq. (5) into Eq. (3), we get the well-known result[11]

$$\langle h_i(t)h_j(t')\rangle = \frac{2k_BT\alpha_0}{\gamma M_s V}\delta_{ij}\delta(t-t'), \tag{6}$$

where i and j denote components orthogonal to the equilibrium magnetization direction. The full random response of the magnetization can be obtained by adding the random field $\mathbf{h}(t)$ to the effective field in the LLG equation.

We now turn our attention to a more complex system, i.e., a metallic ferromagnet in which $\mathbf{m}$ varies along some direction in space, say, the y axis. It is assumed that the spatial variation is adiabatic, i.e., slow on the scale of the ferromagnetic coherence length. The ferromagnet is furthermore assumed to be translationally invariant in the x and z directions, and its magnetization magnitude is taken to be constant and equal to the saturation magnetization M_s. In general, the dynamics and fluctuations of such a magnetization texture depend on position. Due to the spatial variation in the magnetization, *longitudinal* (i.e., polarized parallel to the local magnetization) spin current fluctuations transfer spin angular momentum to the ferromagnet. The resulting enhancement of the magnetization noise is described by introducing a random magnetic field, whose correlator is inhomogenous and anisotropic, in contrast to Eq. (6). By the FDT, the correlator is related to the magnetization damping, which acquires a nonlocal tensor structure. The time scale of electronic motion is much shorter than the typical precession period of magnetization dynamics. This has implicitly been invoked already in Eq. (6). We disregard the effect of spin-flip scattering on the noise properties. Spin-flip corrections in Fe, Ni, and Co are expected to be small because the spin-flip lengths are long compared to the length scale of spatial variation (domain-wall width). Spin-flip is important in Py. However, domain walls in Py are so wide that the effects discussed here are not important anyway.

It is convenient to transform the magnetization texture to a rotated reference frame, defined in terms of the equilibrium (average) magnetization direction $\mathbf{m}_0(y)=\langle\mathbf{m}(y,t)\rangle$ of the texture. The three orthonormal unit vectors spanning this position-dependent frame is $\hat{\mathbf{v}}_1=\hat{\mathbf{v}}_2\times\hat{\mathbf{v}}_3$, $\hat{\mathbf{v}}_2=(d\mathbf{m}_0/dy)/|d\mathbf{m}_0/dy|$, and $\hat{\mathbf{v}}_3=\mathbf{m}_0$. The local gauge,

$$U(y) = [\hat{\mathbf{v}}_1(y) \;\; \hat{\mathbf{v}}_2(y) \;\; \hat{\mathbf{v}}_3(y)]^T, \tag{7}$$

transforms the magnetization, and hence the relevant equations involving the magnetization, to this reference frame. That is, $U\mathbf{m}_0(y)\equiv\tilde{\mathbf{m}}_0=\hat{\mathbf{z}}$, where the tilde indicates a vector in the transformed frame. We note also that $U\hat{\mathbf{v}}_1=\hat{\mathbf{x}}$ and $U\hat{\mathbf{v}}_2=\hat{\mathbf{y}}$, and that U is orthogonal, i.e., $U^{-1}=U^T=[\hat{\mathbf{v}}_1 \;\; \hat{\mathbf{v}}_2 \;\; \hat{\mathbf{v}}_3]$.

We consider a charge current I flowing through the ferromagnet along the y axis. Assuming that the equilibrium magnetization direction $\mathbf{m}_0(y)$ changes adiabatically, the electron spins align with the changing magnetization direction when propagating through the texture. The spin current is then anywhere longitudinal, and hence given by $\mathbf{I}_s(y)=I_s\mathbf{m}_0(y)$. The alignment of the electron spins causes a torque $\boldsymbol{\tau}(y)=d\mathbf{I}_s(y)/dy$ on the ferromagnet. Since $d\mathbf{I}_s(y)/dy$ is perpendicular to $\mathbf{m}_0(y)$, the torque can be written as $\boldsymbol{\tau}(y)=-\mathbf{m}_0(y)\times[\mathbf{m}_0(y)\times d\mathbf{I}_s(y)/dy]$, or as $\tilde{\boldsymbol{\tau}}(y)=U\boldsymbol{\tau}(y)=-\tilde{\mathbf{m}}_0\times[\tilde{\mathbf{m}}_0\times U d\mathbf{I}_s(y)/dy]$ in the local gauge. When $I=0$, which we will take in the following, $I_s=0$ and $\tilde{\boldsymbol{\tau}}=0$ on average. However, at $T\neq 0$ thermal fluctuations of the spin current result in a fluctuating spin-transfer torque,

$$\Delta\tilde{\boldsymbol{\tau}}(y,t) = -\Delta I_s(t)\tilde{\mathbf{m}}_0\times\left[\tilde{\mathbf{m}}_0\times U\frac{d\mathbf{m}_0(y)}{dy}\right], \tag{8}$$

where $\Delta I_s(t)$ are the time-dependent spin current fluctuations with zero mean, propagating along the y direction.

The action of the fluctuating torque on the magnetization is described by adding the term $\gamma\Delta\boldsymbol{\tau}/(M_sA)$ to the right-hand side of the LLG equation. Here A is the cross section (in the xz plane) of the ferromagnetic wire. By linearizing and transforming the LLG equation to the rotated reference frame, the fluctuating torque (8) can be represented by a random magnetic field $\tilde{\mathbf{h}}'(y,t)=\Delta I_s(t)/(M_sA)[\tilde{\mathbf{m}}_0\times U d\mathbf{m}_0(y)/dy]$, analogous to $\mathbf{h}(t)$ discussed above. Using Eq. (7)

$$\tilde{\mathbf{h}}'(y,t) = -\frac{\Delta I_s(t)}{M_sA}\left|\frac{d\mathbf{m}_0(y)}{dy}\right|\hat{x} \tag{9}$$

is seen to be normal to both the current direction and magnetization.

The longitudinal spin current fluctuations $\Delta I_s(t)$ can be found by Landauer-Büttiker scattering theory.[6,12] Disregarding spin-flip processes, the spin-up and spin-down electrons flow in different and independent channels. In the low-frequency regime, in which charge is instantly conserved, longitudinal spin current fluctuations are perfectly correlated throughout the entire ferromagnet. Hence, the thermal spin current fluctuations are given by[6,12]

$$\langle\Delta I_s(t)\Delta I_s(t')\rangle = \frac{\hbar^2}{(2e)^2}2k_BT(G_\uparrow+G_\downarrow)\delta(t-t'), \tag{10}$$

where $G_{\uparrow(\downarrow)}$ is the conductance for electrons with the spin aligned (anti)parallel with the magnetization. This expression is simply the Johnson-Nyquist noise generalized to spin currents.[6] We find from Eqs. (9) and (10)

$$\langle\tilde{h}_x'(y,t)\tilde{h}_x'(y',t')\rangle = \frac{2k_BT\xi_{xx}(y,y')}{\gamma M_s V}\delta(t-t') \tag{11}$$

for the correlator of the current-induced random field, with

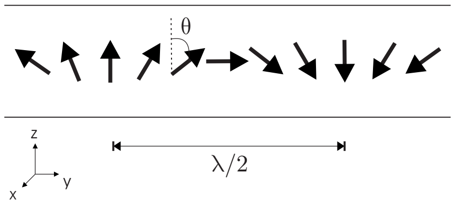

FIG. 1. An example of a nonuniform ferromagnet. The magnetization rotates with wavelength λ in the yz plane, forming a spin spiral.

$$\xi_{xx}(y,y') = \frac{\gamma\hbar^2\sigma}{4e^2M_s}\left|\frac{d\mathbf{m}_0(y)}{dy}\right|\left|\frac{d\mathbf{m}_0(y')}{dy}\right|, \tag{12}$$

and $\sigma=(G_\uparrow+G_\downarrow)L/A$ is the total conductivity. Recall that $\tilde{h}_y'(t)=\tilde{h}_z'(t)=0$. Equation (11) describes the *nonlocal anisotropic* magnetization noise due to thermal current fluctuations in adiabatic nonuniform ferromagnets. This excess noise vanishes with the spatial variation in the magnetization. As a consequence of Eq. (10), the random-field correlator depends nonlocally on the magnetization gradient.

According to the FDT, the thermal noise is related to the magnetization damping. Since the noise correlator (11) is inhomogeneous and anisotropic, the corresponding damping must in general be a nonlocal tensor. To evaluate the damping, we hence need the spatially resolved version of the FDT, which reads

$$\langle \delta\tilde{m}_i(y,t)\delta\tilde{m}_j(y',t')\rangle = \frac{k_BT}{M_sA}\int d\omega e^{-i\omega(t-t')} \times \frac{\chi_{ij}(y,y',\omega)-\chi_{ji}^*(y',y,\omega)}{i2\pi\omega}, \tag{13}$$

in the local gauge. Here $\delta\tilde{\mathbf{m}}(y,t)=U\delta\mathbf{m}(y,t)=\delta m_x(y,t)\hat{\mathbf{x}}+\delta m_y(y,t)\hat{\mathbf{y}}$ are the spatially dependent transformed magnetization fluctuations. Analogous to Eq. (1), the susceptibility is defined as

$$\Delta\tilde{m}_i(y,t) = \sum_j \int\int dy'dt'\chi_{ij}(y,y',t-t')\tilde{h}_j^{(\mathrm{ext})}(y',t'), \tag{14}$$

with transformed external field and magnetization: $\tilde{h}_j^{(\mathrm{ext})}(y,t)=Uh_j^{(\mathrm{ext})}(y,t)$ and $\Delta\tilde{\mathbf{m}}(y,t)=U\Delta\mathbf{m}(y,t)$. The susceptibility in the local gauge frame differs from Eq. (5) and its evaluation is not trivial. It is straightforward to generalize Eqs. (13) and (14) to the case of general three-dimensional dynamics.

We may substitute $\tilde{h}_j^{(\mathrm{ext})}(y',t')$ by $\tilde{h}_j'(y',t')$ in Eq. (14) to find the fluctuations $\delta\tilde{\mathbf{m}}(y,t)$ of the magnetization vector caused by the spin-transfer torque. Combining this expression with Eqs. (13) and (11), we arrive at an integral equation for the unknown susceptibility, from which the nonlocal tensor damping follows. Instead of finding a numerical solution for an arbitrary texture, we consider here a ferromagnetic spin spiral as shown in Fig. 1, for which the description of magnetization noise can be mapped onto the macrospin problem. A simple analytical result can then be found, allowing for a comparison with Eq. (6), and hence an estimate of the relative strength and importance of the current-induced noise and damping.

Spin spirals can be found in some rare-earth metals[13] and in the γ phase of iron,[14] and are described by $\mathbf{m}_0(y)=[0,\sin\,\theta(y),\cos\,\theta(y)]$, where $\theta(y)=2\pi y/\lambda=qy$, with λ the wavelength of the spiral. Then $d\mathbf{m}_0(y)/dy=q[0,\cos\,\theta(y),-\sin\,\theta(y)]$ so that $|d\mathbf{m}_0(y)/dy|=q$. As emphasized earlier, our theory is applicable when the wavelength is much larger than the magnetic coherence length. For transition-metal ferromagnets, the coherence length is of the order of a few ångström. From Eq. (12) we find $\xi_{xx}=\gamma\hbar^2\sigma q^2/(4e^2M_s)$. The current-induced noise correlator (11) for spin spirals is hence homogeneous,

$$\langle\tilde{h}_x'(t)\tilde{h}_x'(t')\rangle = \frac{2k_BT\xi_{xx}}{\gamma M_sV}\delta(t-t'), \tag{15}$$

similar to Eq. (6) but anisotropic. The problem of relating noise to damping in terms of the FDT is therefore isomorphic to the macrospin problem: the transformation (7) can be used to show that equations analogous to Eqs. (1)–(6) are valid for the spin spiral when analyzed in the local gauge frame. It is then seen that the damping term corresponding to Eq. (15) is

$$\tilde{\mathbf{m}} \times \overleftrightarrow{\xi}\frac{d\tilde{\mathbf{m}}}{dt} \tag{16}$$

in the transformed representation. Here

$$\overleftrightarrow{\xi} = \begin{pmatrix} \xi_{xx} & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

is the 2×2 tensor Gilbert damping in the xy plane. Hence, ξ_{xx} is the enhancement of the Gilbert damping caused by the spatial variation in the magnetization and the spin-transfer torque. Due to its anisotropic nature, $\overleftrightarrow{\xi}$ is inside the cross product in Eq. (16), ensuring that the LLG equation preserves the length of the unit magnetization vector $\tilde{\mathbf{m}}$.

To get a feeling for the significance of the current-induced noise and damping, we evaluate $\overleftrightarrow{\xi}$ numerically for a spin spiral with wavelength 20 nm and compare with α_0. Taking parameter values for α_0, M_s, and σ from Refs. 15–18, we find $\xi_{xx}\approx5\alpha_0$ for Fe (with $\alpha_0=0.002$) and $\xi_{xx}\approx4\alpha_0$ for Co (with $\alpha_0=0.005$). Hence, anisotropic current-induced noise and damping in spin spirals can be substantial. Considering half a wavelength of the spin spiral as a simple domain-wall profile, these results furthermore suggest that a significant current-induced magnetization noise and damping should be expected in narrow (width ~10 nm) domain walls in typical transition-metal ferromagnets. However, the curvature of realistic domain-wall profiles differ somewhat from that of spin spirals, especially near the ends of the walls. We may conclude that $\xi_{xx}\approx5\alpha_0$ is an upper bound on the expected inhomogeneous current-induced damping in the center of a 10 nm domain wall in Fe while the damping is substantially less near the ends.

The increased damping should play a central role in, e.g., field-induced motion of narrow domain walls. We predict that the steady-state domain-wall velocity is appreciably reduced. The increased damping is also important in domain-wall motion induced by electric currents. However, in this case, the current-induced nonadiabatic torque[4] can also be significantly enhanced, a complication that is beyond the present paper. We can therefore not predict whether the steady-state current-induced domain-wall motion is lower or higher in narrow walls as compared to wide walls.

The enhancement of the Gilbert damping calculated above has consequences for spin spirals and domain walls. Linear spin waves are not affected to the lowest order in q. While in this paper we focus on the longitudinal spin current noise, there is also a transverse contribution not captured by our analysis, which leads to spin-wave damping proportional to q^2.[19]

So far we have only considered thermal current noise; let us finally turn to shot noise. With the voltage U across the ferromagnet turned on, a nonzero current I flows in the y direction. Disregarding spin-flip processes, the resulting spin current shot noise is[6,12]

$$\langle \Delta I_s^{(\mathrm{sh})}(t) \Delta I_s^{(\mathrm{sh})}(t') \rangle = \frac{\hbar^2}{(2e)^2} eUFG\delta(t-t') \qquad (18)$$

at zero temperature. Here the superscript (sh) emphasizes that we are now looking at shot noise. The Fano factor F is between 0 and 1 for noninteracting electrons.[20] When the length of the metal exceeds the electron–phonon-scattering length λ_{ep}, which is strongly temperature dependent, shot noise vanishes.[12,20] The contribution from shot noise to the magnetization noise is found by replacing Eq. (10) with Eq. (18) in the above calculations. Only at high voltages and low temperatures can shot noise compete with the thermal one. In, e.g., experiments on current-induced domain-wall motion, typical applied current densities are $j \sim 10^8$ A/cm^2,[4] which for a 100-nm-long Fe wire translates into $U=10$ mV. At such high current densities, Joule heating raises the temperature significantly above the ambient one.[21] This reduces the electron–phonon-scattering length, and hence the shot noise, while increasing the thermal noise. As a result, the ratio of shot noise to thermal current noise, $eUF/2k_BT$, will be small in long ferromagnetic wires. We expect shot noise to be more important in, e.g., domain walls that are confined to point contacts with diameter smaller than λ_{ep}.

In summary, we have calculated current-induced magnetization noise and damping in nonuniform ferromagnets. Taking into account both thermal and shot noise, we evaluated the fluctuating spin-transfer torque on the magnetization. The resulting magnetization noise was calculated in terms of a random magnetic field. Employing the FDT, the corresponding enhancement of the Gilbert damping was identified for spin spirals.

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